

RECOVERY OF MISSING SAMPLES IN OVERSAMPLING FORMULAS FOR BAND LIMITED FUNCTIONS

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ABSTRACT. In a previous paper we constructed frames and oversampling formulas for band-limited functions, in the framework of the theory of shift-invariant spaces. In this article we study the problem of recovering missing samples.

We find a sufficient condition for the recovery of a finite set of missing samples. The condition is expressed as a linear independence of the components of a vector W over the space of trigonometric polynomials determined by the frequencies of the missing samples. We apply the theory to the derivative sampling of any order and we illustrate our results with a numerical experiment.

1. INTRODUCTION

A band-limited signal is a function which belongs to the space B_ω of functions in $L^2(\mathbb{R})$ whose Fourier transforms have support in $[-\omega, \omega]$. Functions in this space can be represented by their Whittaker-Kotelnikov-Shannon series, which is the expansion in terms of the orthonormal basis of translates of the *sinc* function. The coefficients of the expansion are the samples of the function at a uniform grid on \mathbb{R} , with “density” ω/π (*Nyquist* density). The sampling theory for band-limited functions stems from this series representation and has been extended to more general expansions, as Riesz bases and frames, formed by the translates of one or more functions.

Let Φ be a finite subset of B_ω and fix a positive real number t_o . If the set of translates $E_{\Phi, t_o} = \{\tau_{kt_o} \varphi, \varphi \in \Phi, k \in \mathbb{Z}\}$ is a frame for B_ω and the set Φ^* of dual functions is known, then any function can be reconstructed via the synthesis formula (2.1). The elements of Φ are called *generators*. The coefficients of the expansion are samples of the convolution of the function with the elements of the dual family E_{Φ^*, t_o} .

If the family is a Riesz basis, the loss of even one sample prevents the reconstruction, since the elements of the family E_{Φ, t_o} are linearly independent. On the contrary, if the family E_{Φ, t_o} is overcomplete, recovery of a finite set of samples is possible under suitable assumptions, because of the redundancy of the system.

This paper deals with the problem of recovering missing samples of band-limited functions. We shall use the frame representation formulas constructed in [DP] in the framework of the theory of frames for shift-invariant spaces.

Key words and phrases. frame, Riesz basis, shift-invariant space, sampling formulas, band-limited functions.

We recall that B_ω is t_o -shift-invariant for any t_o , i.e. is invariant under all translations $\tau_{kt_o}, k \in \mathbb{Z}$, by integer multiples of t_o . A finite subset Φ of a t_o -shift-invariant space S is called a set of generators if S is the closure of the space generated by the family E_{Φ, t_o} . Shift-invariant spaces can have different sets of generators; the smallest number of generators is called the length of the space.

The structure of shift-invariant spaces was first investigated by C. de Boor, R. DeVore [BDR]; successively A. Ron and Z. Shen introduced their Gramian analysis and characterized sets of generators whose translates form Bessel sequences, frames and Riesz bases [RS]. Their conditions are expressed in terms of the eigenvalues of the Gramian matrix. More recently, building on their results, for the space B_ω we have obtained more explicit conditions and explicit formulas for the Fourier transforms of the dual generators [DP]; the same paper contains also multi-channel oversampling formulas for band-limited signals and an application to the first and second derivative oversampling formulas. These are a generalization to frames of the classical derivative sampling formulas, where the family E_{Φ, t_o} is a Riesz basis (see [Hi]). The coefficients of the expansion are the values of the function and its derivatives at the sample points.

In this paper we assume that E_{Φ, t_o} is a frame for B_ω and we find sufficient conditions for the recovery of a finite set of missing samples. This problem has been investigated by P.J.S.G Ferreira in [F], where it is shown that, in the case of a generalized Kramer sampling, under suitable oversampling assumptions, any finite set of samples can be recovered from the others. Successively D.M.S. Santos and Ferreira [SF] considered the case of a particular two-channel derivative oversampling formula [SF]. The work of Santos and Ferreira has been generalized to arbitrary two-channels by Y. M. Hong and K. H. Kwon [KK].

In this paper we consider general frames E_{Φ, t_o} of B_ω where the number of elements of Φ is minimal, i.e. equal to the length of B_ω as t_o -shift invariant space, and find sufficient conditions for the recovery of any finite subset of missing samples.

The paper is organized as follows. In Section 2 we introduce the pre-Gramian and Gramian matrices and recall some results, proved in [DP], that characterize sets of generators whose translates form frames and Riesz bases, in terms of the Fourier transforms of the generators.

In Section 3, we fix the parameter t_o and we consider a family $\Phi = \{\varphi_1, \varphi_2, \dots, \varphi_N\}$ of N generators, where N is the length of B_ω as a t_o -shift-invariant space. If the family E_{Φ, t_o} is a frame and not a Riesz basis, we find sufficient conditions to reconstruct any finite set of missing samples (see Theorems 3.4 and 3.5). This result makes use of the particular structure of the mixed-Gramian matrix G_{Φ, Φ^*, t_o} associated to Φ and to the dual generators Φ^* . Roughly speaking, the interval $[0, 2\pi/t_o]$ is the union of two sets, where G_{Φ, Φ^*, t_o} is either the $N \times N$ identity matrix or $I - P_W$, where P_W is the orthogonal projection on a vector W in C^N , which is the cross product of $N - 1$ translates of the Fourier transform of the generators Φ (see Propositions 3.1 and 3.2). The recovery condition is expressed as a linear independence of the components of the vector W over the space of trigonometric polynomials determined by the frequencies of the missing samples. In Section 4 we find families of derivative frames of any order and apply the results of Section 3 to them, showing that it is possible to recover any finite set of missing samples.

2. PRELIMINARIES

In this section we establish notation and we collect some results on frames for shift-invariant spaces. The Fourier transform of a function f in $L^1(\mathbb{R})$ is

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int f(t) e^{-it\xi} dt.$$

The convolution of two functions f and g is $f * g(x) = \int f(x-y)g(y) dy$, so that $\widehat{f * g} = \sqrt{2\pi} \widehat{f} \widehat{g}$. Let h be a positive real number; L_h^p is the space of h -periodic functions on \mathbb{R} such that

$$\|f\|_{L_h^p} = \left(\frac{1}{h} \int_0^h |f(x)|^p dx \right)^{1/p} < \infty.$$

In this paper vectors in \mathbb{C}^N are to be considered as column-vectors; however, to save space, we shall write $x = (x_1, x_2, \dots, x_N)$ to denote the column-vector whose components are x_1, \dots, x_N . With the symbol $\ell^2(\mathbb{Z}; \mathbb{C}^N)$ we shall denote the space of square summable \mathbb{C}^N -valued sequences $c = (c(n))_{n \in \mathbb{Z}}$.

Let H be a closed subspace of $L^2(\mathbb{R})$. Given a subset $\Phi = \{\varphi_j, j = 1, \dots, N\}$ of H and a positive number t_o , denote by E_{Φ, t_o} the set

$$E_{\Phi, t_o} = \{\tau_{nt_o} \varphi_j, n \in \mathbb{Z}, j = 1, \dots, N\};$$

here $\tau_a f(x) = f(x+a)$. The family E_{Φ, t_o} is a frame for H if the operator $T_{\Phi, t_o} : \ell^2(\mathbb{Z}; \mathbb{C}^N) \rightarrow H$ defined by $T_{\Phi, t_o} c = \sum_{j=1}^N \sum_{n \in \mathbb{Z}} c_j(n) \tau_{nt_o} \varphi_j$ is continuous, surjective and $\text{ran}(T_{\Phi, t_o})$ is closed. It is well known that E_{Φ, t_o} is a frame for H if and only if there exist two constants $0 < A \leq B$ such that

$$A\|f\|^2 \leq \sum_{j=1}^N \sum_{n \in \mathbb{Z}} |\langle f, \tau_{nt_o} \varphi_j \rangle|^2 \leq B\|f\|^2 \quad \forall f \in H.$$

The constants A and B are called frame bounds. Denote by $T_{\Phi, t_o}^* : H \rightarrow \ell^2(\mathbb{Z}; \mathbb{C}^N)$ the adjoint of T_{Φ, t_o} . The operator $T_{\Phi, t_o} T_{\Phi, t_o}^* : H \rightarrow H$ is called *frame operator*. Denote by Φ^* the family $\{\varphi_j^*, j = 1, \dots, N\}$, where

$$\varphi_j^* = (T_{\Phi, t_o} T_{\Phi, t_o}^*)^{-1} \varphi_j \quad 1 \leq j \leq N.$$

If E_{Φ, t_o} is a frame for H then E_{Φ^*, t_o} is also a frame, called the *dual frame*, and $T_{\Phi, t_o} T_{\Phi^*, t_o}^* = T_{\Phi^*, t_o} T_{\Phi, t_o}^* = I$. Explicitly

$$(2.1) \quad f = \sum_{j=1}^N \sum_{n \in \mathbb{Z}} \langle f, \tau_{nt_o} \varphi_j^* \rangle \tau_{nt_o} \varphi_j = \sum_{j=1}^N \sum_{n \in \mathbb{Z}} \langle f, \tau_{nt_o} \varphi_j \rangle \tau_{nt_o} \varphi_j^*$$

$\forall f \in H$. Note that

$$(2.2) \quad \langle f, \tau_{nt_o} g \rangle = f * \tilde{g}(-nt_o) \quad \forall n \in \mathbb{Z}$$

where \tilde{g} denotes the function $\tilde{g} = \overline{g(-t)}$. Thus the coefficients of the expansion are the samples of the function $f * \tilde{g}$ in $-nt_o$. The elements of Φ are called *generators* and the elements of Φ^* *dual generators*. If the family E_{Φ, t_o} is a frame for H and the operator T_{Φ, t_o} is injective, then E_{Φ, t_o} is called a *Riesz basis*.

In what follows t_o is a positive parameter. To simplify notation, throughout the paper we shall set

$$h = \frac{2\pi}{t_o}.$$

A subspace S of $L^2(\mathbb{R})$ is t_o -shift-invariant if it is invariant under all translations by a multiple of t_o . The following bracket product plays an important role in Ron and Shen's analysis of shift-invariant spaces. For f and $g \in L^2(\mathbb{R})$, define

$$[f, g] = h \sum_{j \in \mathbb{Z}} f(\cdot + jh) \bar{g}(\cdot + jh).$$

Note that $[f, g]$ is in L_h^1 and $\|[f, f]\|_{L_h^1} = \|f\|_2^2$. The Fourier coefficients of $[\hat{f}, \hat{g}]$ are given by

$$\begin{aligned} [\hat{f}, \hat{g}](n) &= \int_0^h \sum_j \tau_{jh}(\hat{f} \bar{\hat{g}})(x) e^{-2\pi i n \frac{x}{h}} dx = \langle f, \tau_{nt_o} g \rangle \\ (2.3) \quad &= f * \tilde{g}(-nt_o) \quad \forall n \in \mathbb{Z}. \end{aligned}$$

If S is a t_o -shift-invariant space and there exists a finite family Φ such that S is the closed linear span of E_{Φ, t_o} , then we say that S is finitely generated. Riesz bases for finitely generated shift-invariant spaces have been studied by various authors. In [BDR] the authors give a characterization of such bases. A characterization of frames and tight frames also for countable sets Φ has been given by Ron and Shen in [RS]. The principal notions of their theory are the *pre-Gramian*, the *Gramian* and the *dual Gramian* matrices. The *pre-Gramian* J_{Φ, t_o} is the h -periodic function mapping \mathbb{R} to the space of $\infty \times N$ -matrices defined on $[0, h]$ by

$$(2.4) \quad (J_{\Phi, t_o})_{j\ell}(x) = \sqrt{h} \widehat{\varphi_\ell}(x + jh), \quad j \in \mathbb{Z}, \ell = 1, \dots, N.$$

The pre-Gramian J_{Φ, t_o} should not be confused with the matrix-valued function whose entries are $\sqrt{h} \widehat{\varphi_\ell}(x + jh)$, for all $x \in \mathbb{R}$, which is not periodic. Denote by J_{Φ, t_o}^* the adjoint of J_{Φ, t_o} . The $N \times N$ Gramian matrix $G_{\Phi, \Phi^*, t_o} = J_{\Phi, t_o}^* J_{\Phi, t_o}$ plays a crucial role in the recovery of missing samples; its elements are the h -periodic functions

$$(2.5) \quad (G_{\Phi, \Phi^*, t_o})_{j\ell} = [\widehat{\varphi_\ell^*}, \widehat{\varphi_j}].$$

We set

$$\ell = \left\lceil \frac{\omega}{h} \right\rceil + 1,$$

where $[a]$ denotes the greatest integer less than a . In [DP, Corollary 2.3] we found the length of B_ω , as t_o -shift-invariant space:

$$(2.6) \quad \text{len}_{t_o}(B_\omega) = \begin{cases} 2\ell & \text{if } \frac{\omega}{\ell} \leq h < \frac{\omega}{\ell - \frac{1}{2}}, \\ 2\ell - 1 & \text{if } \frac{\omega}{\ell - \frac{1}{2}} \leq h < \frac{\omega}{\ell - 1}. \end{cases}$$

We also gave necessary and sufficient conditions for E_{Φ, t_o} to be a Riesz basis or a frame for B_ω [DP, Theorems 3.6, 3.7]. The result is based on the analysis of the structure of the matrix J_{Φ, t_o} ; we restate it below in Theorems 2.1 and 2.2 for the sake of the reader.

Strictly speaking, the pre-Gramian J_{Φ, t_o} is an infinite matrix. However, in [DP, Lemma 3.3] it has been shown that if Φ is a set of generators of B_ω , then all but a finite number of the rows of J_{Φ, t_o} vanish; hence we may identify it with a finite matrix. Indeed, consider separately the two cases $\omega/\ell \leq h < \omega/(\ell - 1/2)$ and $\omega/(\ell - 1/2) \leq h < \omega/(\ell - 1)$.

Assume first that $\omega/\ell \leq h < \omega/(\ell - 1/2)$; then $0 \leq -\omega + \ell h < \omega - (\ell - 1)h < h$. We denote by I_-, I, I_+ the intervals defined by

$$I_- = (0, -\omega + \ell h), \quad I = (-\omega + \ell h, \omega - (\ell - 1)h), \quad I_+ = (\omega - (\ell - 1)h, h).$$

By (2.6) $\text{len}_{t_o}(B_\omega) = 2\ell$; if $\Phi = \{\varphi_j : 1 \leq j \leq 2\ell\}$ is a subset of B_ω of cardinality 2ℓ , by [DP, Lemma 3.3] all the rows of the matrix J_{Φ, t_o} vanish, except possibly the rows $(\tau_{jh}\widehat{\varphi}_1, \tau_{jh}\widehat{\varphi}_2, \dots, \tau_{jh}\widehat{\varphi}_{2\ell})$, $-\ell \leq j \leq \ell - 1$. Thus we identify the infinite matrices J_{Φ, t_o} , J_{Φ^*, t_o} , their adjoints and the matrices G_{Φ, t_o} , G_{Φ, Φ^*, t_o} with their $2\ell \times 2\ell$ submatrices corresponding to these rows. The i -th column of J_{Φ, t_o} , $1 \leq i \leq 2\ell$ is

$$(2.7) \quad \sqrt{h} \begin{bmatrix} 0 \\ \tau_{-(\ell-1)h}\widehat{\varphi}_i \\ \vdots \\ \widehat{\varphi}_i \\ \vdots \\ \tau_{(\ell-2)h}\widehat{\varphi}_i \\ \tau_{(\ell-1)h}\widehat{\varphi}_i \end{bmatrix} \text{ in } I_- \quad \sqrt{h} \begin{bmatrix} \tau_{-\ell h}\widehat{\varphi}_i \\ \tau_{-(\ell-1)h}\widehat{\varphi}_i \\ \vdots \\ \widehat{\varphi}_i \\ \vdots \\ \tau_{(\ell-2)h}\widehat{\varphi}_i \\ \tau_{(\ell-1)h}\widehat{\varphi}_i \end{bmatrix} \text{ in } I \quad \sqrt{h} \begin{bmatrix} \tau_{-\ell h}\widehat{\varphi}_i \\ \tau_{-(\ell-1)h}\widehat{\varphi}_i \\ \vdots \\ \widehat{\varphi}_i \\ \vdots \\ \tau_{(\ell-2)h}\widehat{\varphi}_i \\ 0 \end{bmatrix} \text{ in } I_+$$

(see Lemma 3.3 in [DP]). The same formulas hold for the matrix J_{Φ^*, t_o} , with φ replaced by φ^* . We note that the matrices G_{Φ, t_o} and J_{Φ, t_o} have the same rank. Let A be a $n \times m$ matrix with complex entries, $n \leq m$; we shall denote by $\|A\|$ the norm of A as linear operator from \mathbb{C}^m to \mathbb{C}^n and by $[A]_n$ the sum of the squares of the absolute values of the minors of order n of A .

Theorem 2.1. [DP, Theorem 3.6] *Suppose that $\omega/\ell \leq h < \omega/(\ell - 1/2)$. Let $\Phi = \{\varphi_j : 1 \leq j \leq 2\ell\}$ be a subset of B_ω . Then E_{Φ, t_o} is a frame for B_ω if and only if there exist positive constants δ, γ, σ and η such that*

$$(2.8) \quad \delta \leq \sum_{j=1}^{2\ell} |\widehat{\varphi}_j|^2 \leq \gamma \quad \text{a.e. in } (-\omega, \omega),$$

$$(2.9) \quad [J_{\Phi, t_o}]_{2\ell-1} \geq \sigma \quad \text{a.e. in } I_- \cup I_+,$$

$$(2.10) \quad |\det J_{\Phi, t_o}| \geq \eta \quad \text{a.e. in } I.$$

If $h = \omega/\ell$ the intervals I_- and I_+ are empty. In this case E_{Φ, t_o} is a Riesz basis for B_ω if and only if conditions (2.8) and (2.10) hold.

Next we consider the case $\omega/(\ell - 1/2) \leq h < \omega/(\ell - 1)$. Then $0 < \omega - (\ell - 1)h \leq -\omega + \ell h < h$; in this case we denote by K_-, K, K_+ the intervals defined by

$$(2.11) \quad K_- = (0, \omega - (\ell - 1)h), \quad K = (\omega - (\ell - 1)h, -\omega + \ell h), \quad K_+ = (-\omega + \ell h, h).$$

By (2.6) $\text{len}_{t_o}(B_\omega) = 2\ell - 1$. Let $\Phi = \{\varphi_j : 1 \leq j \leq 2\ell - 1\}$ be a subset of B_ω of cardinality $2\ell - 1$. By [DP, Lemma 3.3] all the rows of the matrix J_{Φ, t_o} , except possibly $(\tau_{jh}\widehat{\varphi}_1, \tau_{jh}\widehat{\varphi}_2, \dots, \tau_{jh}\widehat{\varphi}_{2\ell})$, $-\ell \leq j \leq \ell - 1$, vanish. Thus we identify the infinite matrices J_{Φ, t_o} , J_{Φ^*, t_o} , their adjoints and the matrices G_{Φ, t_o} , G_{Φ, Φ^*, t_o} with

their $2\ell - 1 \times 2\ell - 1$ submatrices corresponding to these rows. The i -th column of J_{Φ, t_o} , $1 \leq i \leq 2\ell - 1$ is

$$(2.12) \quad \sqrt{h} \begin{bmatrix} \tau_{-(\ell-1)h} \hat{\varphi}_i \\ \vdots \\ \hat{\varphi}_i \\ \vdots \\ \tau_{(\ell-2)h} \hat{\varphi}_i \\ \tau_{(\ell-1)h} \hat{\varphi}_i \end{bmatrix} \text{ in } K_- \quad \sqrt{h} \begin{bmatrix} \tau_{-(\ell-1)h} \hat{\varphi}_i \\ \vdots \\ \hat{\varphi}_i \\ \vdots \\ \tau_{(\ell-2)h} \hat{\varphi}_i \\ 0 \end{bmatrix} \text{ in } K \quad \sqrt{h} \begin{bmatrix} \tau_{-\ell h} \hat{\varphi}_i \\ \tau_{-(\ell-1)h} \hat{\varphi}_i \\ \vdots \\ \hat{\varphi}_i \\ \vdots \\ \tau_{(\ell-2)h} \hat{\varphi}_i \end{bmatrix} \text{ in } K_+.$$

Theorem 2.2. [DP, Theorem 3.7] *Suppose that $\omega/(\ell - 1/2) \leq h < \omega/(\ell - 1)$, $\ell \neq 1$. Let $\Phi = \{\varphi_j : 1 \leq j < 2\ell - 1\}$ be a subset of B_ω . Then E_{Φ, t_o} is a frame for B_ω if and only if there exist positive constants δ, γ, σ and η such that*

$$(2.13) \quad \delta \leq \sum_{j=1}^{2\ell-1} |\hat{\varphi}_j|^2 \leq \gamma \quad \text{a.e. in } (-\omega, \omega),$$

$$(2.14) \quad [J_{\Phi, t_o}]_{2\ell-2} \geq \sigma \quad \text{a.e. in } K,$$

$$(2.15) \quad |\det J_{\Phi, t_o}| \geq \eta \quad \text{a.e. in } K_- \cup K_+.$$

If $h = \omega/(\ell - \frac{1}{2})$ the interval K is empty. In this case E_{Φ, t_o} is a Riesz basis if and only if conditions (2.13) and (2.15) hold.

3. RECOVERY OF MISSING SAMPLES

In this section we consider the problem of reconstructing a band-limited function when a finite set of samples is missing. Let t_o be a positive real number and L the length of B_ω as t_o -shift-invariant space. Let $\Phi = \{\varphi_1, \dots, \varphi_L\}$ be a set of functions in B_ω such that E_{Φ, t_o} is a frame for B_ω . Then any function in B_ω can be reconstructed via the syntesis formula (2.1). We recall that the coefficients of the expansion are the samples of the functions $f_j = f * \tilde{\varphi}_j$ at integer multiples of t_o and we may rewrite (2.1) as

$$(3.1) \quad f = \sum_{j=1}^L \sum_{n \in \mathbb{Z}} f_j(nt_o) \tau_{-nt_o} \varphi_j^* \quad \forall f \in B_\omega.$$

The recovery of lost samples in band-limited signals has already been investigated by Ferreira for the classical (one-channel) Shannon formula [F], and by Santos and Ferreira for a particular two-channel derivative sampling [SF]. In the latter article, the authors show that a finite number of missing samples of the function or its derivative can be recovered. In [KK] Hong and Kwon have generalized these results to a two-channel sampling formula, finding sufficient conditions for the recovery of missing samples. In both papers [SF, KK], the authors work with particular reconstruction formulas, obtained by projecting Riesz basis generators of B_ω and their duals into the space B_{ω_a} with $\omega_a < \omega$. With this technique the projected family is a frame; note that projecting the dual of a Riesz basis does not yield

the canonical dual. In particular, the coefficients of the expansion of a function computed with respect to the projected duals are not minimal in the least square norm.

We find sufficient conditions such that, if E_{Φ, t_o} is a frame (and not a Riesz basis), any finite set of missing samples may be recovered (see Theorems 3.4 and 3.5 below).

In the recovery of missing samples the structure of the mixed Gramian matrix $G_{\Phi, \Phi^*, t_o} = J_{\Phi, t_o}^* J_{\Phi^*, t_o}$ plays a crucial role. Propositions 3.1 and 3.2 below show that, a.e. in the interval $[0, h]$, the matrix $I - G_{\Phi, \Phi^*, t_o}$ is either the identity or a projection on a subspace of codimension one. To prove them, we recall some matrix identities obtained in [DP]. By formula (4.2) in [DP]

$$(3.2) \quad J_{\Phi, t_o} = J_{\Phi, t_o} J_{\Phi, t_o}^* J_{\Phi^*, t_o} = J_{\Phi, t_o} G_{\Phi, \Phi^*, t_o}.$$

Under the assumptions of Theorems 2.1 or 2.2, the interval $[0, h]$ is the disjoint union of three intervals where the pre-Gramian is either invertible or has rank $L-1$. Where the pre-Gramian is invertible one can solve for J_{Φ^*, t_o} in (3.2), obtaining that

$$(3.3) \quad J_{\Phi^*, t_o} = (J_{\Phi, t_o}^*)^{-1}.$$

In the intervals where the pre-Gramian has rank $L-1$, either its first or last row vanishes; this happens in the intervals $I_- \cup I_+$ in formula (2.7) and in the interval K in (2.12). We shall denote by \mathbb{J}_{Φ, t_o} and \mathbb{J}_{Φ^*, t_o} the $(L-1) \times L$ submatrices of J_{Φ, t_o} and J_{Φ^*, t_o} obtained by deleting the vanishing row. In this case $G_{\Phi, \Phi^*, t_o} = \mathbb{J}_{\Phi, t_o}^* \mathbb{J}_{\Phi^*, t_o}$. Since \mathbb{J}_{Φ^*, t_o} is the Moore-Penrose inverse $(\mathbb{J}_{\Phi, t_o}^*)^\dagger$ of \mathbb{J}_{Φ, t_o}^* , we have that

$$(3.4) \quad G_{\Phi, \Phi^*, t_o} = \mathbb{J}_{\Phi, t_o}^* (\mathbb{J}_{\Phi^*, t_o})^\dagger.$$

We refer the reader to [BIG] for the definition and the properties of the Moore-Penrose inverse of a matrix.

We shall denote by W the cross product of the rows of the matrix $h^{-1/2} \mathbb{J}_{\Phi, t_o}$. We observe that, if $\omega/(\ell-1/2) < h < \omega/(\ell-1)$, $\ell \neq 1$, then

$$W = \bigtimes_{j=-\ell+1}^{\ell-2} \tau_{jh} \widehat{\Phi} \quad \text{a.e. in } K;$$

while, if $\omega/\ell < h < \omega/(\ell-1/2)$, then

$$W = \bigtimes_{j=-\ell+1}^{\ell-1} \tau_{jh} \widehat{\Phi} \quad \text{a.e. in } I_- \quad W = \bigtimes_{j=-\ell}^{\ell-2} \tau_{jh} \widehat{\Phi} \quad \text{a.e. in } I_+.$$

Observe that, since W is orthogonal to the rows of \mathbb{J}_{Φ, t_o} and $\dim \text{Ker } \mathbb{J}_{\Phi, t_o} = 1$, then $\text{Ker } \mathbb{J}_{\Phi, t_o} = \text{span}(W)$.

We denote by I_n the $n \times n$ identity matrix. Given a vector $v \in \mathbb{C}^n$, we denote by P_v the orthogonal projection on v in \mathbb{C}^n .

Proposition 3.1. *Let $\omega/(\ell-1/2) < h < \omega/(\ell-1)$, $\ell \neq 1$ and let $\Phi = \{\varphi_j, 1 \leq j \leq 2\ell-1\}$ be a subset of B_ω . If E_{Φ, t_o} is a frame for B_ω then*

$$(3.5) \quad G_{\Phi, \Phi^*, t_o} = I_{2\ell-1} \quad \text{a.e. in } K_- \cup K_+,$$

$$(3.6) \quad G_{\Phi, \Phi^*, t_o} = I_{2\ell-1} - P_W \quad \text{a.e. in } K.$$

Proof. By (2.15), the pre-Gramian J_{Φ, t_o}^* is invertible in the intervals K_- and K_+ ; hence formula (3.5) follows from (3.2). Consider now the interval K . By (3.4) and the properties of the Moore-Penrose inverse, G_{Φ, Φ^*, t_o} is the orthogonal projection on $\text{Ran}(\mathbb{J}_{\Phi, t_o}^*)$ in $\mathbb{C}^{2\ell-1}$. On the other hand, since \mathbb{J}_{Φ, t_o} has maximum rank,

$$\text{span}(W) = \text{Ker}(\mathbb{J}_{\Phi, t_o}) = \text{Ran}(\mathbb{J}_{\Phi, t_o}^*)^\perp.$$

Hence the projection on $\text{Ran}(\mathbb{J}_{\Phi, t_o}^*)$ is $I_{2\ell-2} - P_W$. This proves (3.6). \square

A similar argument yields

Proposition 3.2. *Let $\omega/\ell < h < \omega/(\ell - 1/2)$ and let $\Phi = \{\varphi_j, 1 \leq j \leq 2\ell\}$ be a subset of B_ω . If E_{Φ, t_o} is a frame for B_ω , then*

$$(3.7) \quad G_{\Phi, \Phi^*, t_o} = I_{2\ell} \quad \text{a.e. in } I$$

$$(3.8) \quad G_{\Phi, \Phi^*, t_o} = I_{2\ell} - P_W \quad \text{a.e. in } I_- \cup I_+.$$

Let $\mathcal{I} = \{\ell_1, \ell_2, \dots, \ell_N\}$ be a given set of integers. We give sufficient conditions to reconstruct the missing samples

$$(3.9) \quad f_j(nt_o) \quad n \in \mathcal{I}, \quad 1 \leq j \leq L.$$

Theorem 3.4 deals with the case $\omega/(\ell - 1/2) < h < \omega/(\ell - 1)$, i.e. $L = 2\ell - 1$, while Theorem 3.5 deals with the case $\omega/\ell < h < \omega/(\ell - 1/2)$, i.e. $L = 2\ell$.

The sufficient conditions are expressed as the linear independence of the components of the vector W over the set $P_{\mathcal{I}, h}$ of trigonometric polynomials of the form

$$p(t) = \sum_{j=1}^N x_j e^{-\frac{2\pi}{h} i \ell_j t} \quad x_j \in \mathbb{C}.$$

Definition 3.3. *A finite family $\{F_j, j = 1, \dots, m\}$, $m \in \mathbb{N}$, of functions is $P_{\mathcal{I}, h}$ -linearly dependent on a set $\Omega \subset [0, h]$ if there exist non-zero trigonometric polynomials p_1, p_2, \dots, p_m , in $P_{\mathcal{I}, h}$ such that*

$$(3.10) \quad \sum_{k=1}^m p_k(t) F_k(t) = 0 \quad \text{for a.e. } t \in \Omega$$

If $\mathbf{x} = (x_1, x_2, \dots, x_N)$ is a vector in \mathbb{C}^N , we shall denote by $\widehat{\mathbf{x}}$ the trigonometric polynomial in $P_{\mathcal{I}, h}$

$$(3.11) \quad \widehat{\mathbf{x}}(t) = \sum_{j=1}^N x_j e^{-\frac{2\pi}{h} i \ell_j t} \quad t \in \mathbb{R}.$$

If $X = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_L)$ with $\mathbf{x}_j \in \mathbb{C}^N$ for $j = 1, \dots, L$, we write

$$(3.12) \quad \widehat{X}(t) = (\widehat{\mathbf{x}}_1(t), \widehat{\mathbf{x}}_2(t), \dots, \widehat{\mathbf{x}}_L(t)) \quad t \in \mathbb{R}.$$

Thus $X \mapsto \widehat{X}(t)$ maps vectors in $\mathbb{C}^{N \times L}$ to vectors in \mathbb{C}^L for a.e. $t \in \mathbb{R}$.

Theorem 3.4. *Suppose that the assumptions of Theorem 2.2 are satisfied with $h \neq \omega/(\ell - 1/2)$. Let $\mathcal{I} \subset \mathbb{Z}$ be an assigned finite set. If the components of the vector W are $P_{\mathcal{I}, h}$ -independent on K , then for every $f \in B_\omega$ it is possible to recover the samples $f_j(nt_o)$, $n \in \mathcal{I}$, $1 \leq j \leq 2\ell - 1$.*

Proof. Let $\mathcal{I} = \{\ell_1, \ell_2, \dots, \ell_N\}$. By convolving with $\tilde{\varphi}_k$ both sides of expansion formula (3.1), where $L = 2\ell - 1$, and evaluating in $\ell_m t_o$, we get

$$f_k(\ell_m t_o) = \sum_{j=1}^{2\ell-1} \sum_{n \in \mathbb{Z}} f_j(n t_o) (\varphi_j^* * \tilde{\varphi}_k)(\ell_m t_o - n t_o)$$

$k = 1, \dots, 2\ell - 1$, $m = 1 \dots, N$. Next we isolate in the left hand side the terms containing the unknown samples

$$(3.13) \quad f_k(\ell_m t_o) - \sum_{j=1}^{2\ell-1} \sum_{p=1}^N f_j(\ell_p t_o) (\varphi_j^* * \tilde{\varphi}_k)(\ell_m t_o - \ell_p t_o)$$

$$= \sum_{j=1}^{2\ell-1} \sum_{n \notin \mathcal{I}} f_j(n t_o) (\varphi_j^* * \tilde{\varphi}_k)(\ell_m t_o - n t_o)$$

$k = 1, \dots, 2\ell - 1$, $1 \leq m \leq N$. This is a system of $N(2\ell - 1)$ equations in the $N(2\ell - 1)$ unknowns $f_k(\ell_m t_o)$. To write it in a more compact form, we denote the unknowns by $x_j(m)$ and the right hand side of equation (3.13) by $b_k(m)$, i.e.

$$x_k(m) = f_k(\ell_m t_o)$$

$$b_k(m) = \sum_{j=1}^{2\ell-1} \sum_{n \notin \mathcal{I}} f_j(n t_o) (\varphi_j^* * \tilde{\varphi}_k)(\ell_m t_o - n t_o),$$

for $1 \leq k \leq 2\ell - 1$, $1 \leq m \leq N$. Next, we write

$$\mathbf{x}_k = (x_k(1), \dots, x_k(N)), \quad \mathbf{b}_k = (b_k(1), \dots, b_k(N))$$

and

$$X = (\mathbf{x}_1, \dots, \mathbf{x}_{2\ell-1}), \quad B = (\mathbf{b}_1, \dots, \mathbf{b}_{2\ell-1}).$$

We introduce the block matrix

$$(3.14) \quad S = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1(2\ell-1)} \\ S_{21} & S_{22} & \cdots & S_{2(2\ell-1)} \\ \vdots & \vdots & \ddots & \vdots \\ S_{(2\ell-1)1} & S_{(2\ell-1)2} & \cdots & S_{(2\ell-1)(2\ell-1)} \end{bmatrix}$$

where the S_{kj} , $1 \leq j, k \leq 2\ell - 1$, are the submatrices whose entries are

$$(3.15) \quad S_{kj}(m, p) = (\varphi_j^* * \tilde{\varphi}_k)(\ell_m t_o - \ell_p t_o) \quad m, p = 1, \dots, N.$$

Thus the equations in (3.13) can be written

$$(3.16) \quad (I - S)X = B$$

where X and B are vectors in $\mathbb{C}^{N(2\ell-1)}$ and I is the $N(2\ell - 1) \times N(2\ell - 1)$ identity matrix. Thus the missing samples can be recovered if equation (3.16) can be solved for all B , and this happens if and only if 1 is not an eigenvalue of S .

We shall prove that if 1 is an eigenvalue of S , then the components of the vector W are $P_{\mathcal{I}, h}$ -dependent on K ; thereby contradicting the assumptions. Consider the quadratic form $X^* X - X^* S X$, where $*$ denotes conjugate transpose. We claim that for all $X \in \mathbb{C}^{N(2\ell-1)}$

$$(3.17) \quad X^* X - X^* S X = \frac{1}{h} \int_K \left\| \left(I_{2\ell-1} - G_{\Phi, \Phi^*, t_o}(t) \right) \hat{X}(t) \right\|_{\mathbb{C}^{2\ell-1}}^2 dt,$$

where \widehat{X} is given by (3.12) with $L = 2\ell - 1$. Indeed, by (2.3), the (m, p) -entry of the matrix S_{kj} in (3.15) is

$$S_{kj}(m, p) = \frac{1}{h} \int_0^h [\widehat{\varphi}_j^*, \widehat{\varphi}_k](t) e^{i(\ell_m - \ell_p)t_o t} dt \quad m, p = 1, \dots, N.$$

Hence

$$\begin{aligned} X^* S X &= \sum_{j,k=1}^{2\ell-1} \sum_{m,p=1}^N \bar{x}_j(m) S_{j,k}(m, p) x_k(p) \\ &= \frac{1}{h} \sum_{j,k=1}^{2\ell-1} \sum_{m,p=1}^N \bar{x}_j(m) \int_0^h [\widehat{\varphi}_j^*, \widehat{\varphi}_k](t) e^{i(\ell_m - \ell_p)t_o t} dt x_k(p) \\ &= \frac{1}{h} \int_0^h \sum_{j,k=1}^{2\ell-1} \sum_{m=1}^N \bar{x}_j(m) e^{i\ell_m t_o t} [\widehat{\varphi}_j^*, \widehat{\varphi}_k](t) \sum_{p=1}^N x_k(p) e^{-i\ell_p t_o t} dt \\ &= \frac{1}{h} \int_0^h \widehat{X}^*(t) G_{\Phi, \Phi^*, t_o}(t) \widehat{X}(t) dt. \end{aligned}$$

The last identity follows by (2.5). Thus, by the Parseval identity

$$X^* X = \frac{1}{h} \int_0^h \widehat{X}^*(t) \widehat{X}(t) dt$$

and the fact that $G_{\Phi, \Phi^*, t_o}(t)$ is an orthogonal projection for almost every $t \in [0, h]$, we obtain

$$\begin{aligned} X^* X - X^* S X &= \frac{1}{h} \int_0^h \widehat{X}^*(t) (I_{2\ell-1} - G_{\Phi, \Phi^*, t_o}(t)) \widehat{X}(t) dt \\ &= \frac{1}{h} \int_0^h \widehat{X}^*(t) (I_{2\ell-1} - G_{\Phi, \Phi^*, t_o}(t))^2 \widehat{X}(t) dt \\ &= \frac{1}{h} \int_0^h \left\| (I_{2\ell-1} - G_{\Phi, \Phi^*, t_o}(t)) \widehat{X}(t) \right\|_{\mathbb{C}^{2\ell-1}}^2 dt. \end{aligned}$$

Formula (3.17) follows by observing that, by (3.5), the integrand is zero in $K_- \cup K_+ = [0, h] \setminus K$.

Assume that 1 is an eigenvalue of the matrix S . Then, by (3.17), there is a vector $X^o = (\mathbf{x}_1^o, \dots, \mathbf{x}_{2\ell-1}^o)$ in $\mathbb{C}^{N(2\ell-1)}$, $X^o \neq 0$, such that

$$(3.18) \quad (I_{2\ell-1} - G_{\Phi, \Phi^*, t_o}(t)) \widehat{X}^o(t) = 0 \quad \text{for a.e. } t \in K.$$

Since $I_{2\ell-1} - G_{\Phi, \Phi^*, t_o} = P_W$ a.e. in K , by Proposition 3.1, the vectors $\widehat{X}^o(t)$ and $W(t)$ are orthogonal for a.e. t in K , i.e.

$$\sum_{j=1}^{2\ell-1} \widehat{\mathbf{x}}_j^o(t) W_j(t) = 0 \quad \text{for a.e. } t \text{ in } K.$$

Since $\widehat{\mathbf{x}}_j^o(t) \in P_{\mathcal{I}, h}$, $1 \leq j \leq 2\ell - 1$, this implies that the components of W are $P_{\mathcal{I}, h}$ -dependent. \square

Remark. If the matrix S is Hermitian, the condition in the previous theorem is also necessary. Indeed, assume that the components of W are $P_{\mathcal{I}, h}$ -linearly dependent on the interval K . Then there exists a non-null vector $X^o = (\mathbf{x}_1^o, \mathbf{x}_2^o, \dots, \mathbf{x}_{2\ell-1}^o)$

in $\mathbb{C}^{N(2\ell-1)}$ such that \widehat{X}^o is orthogonal to W in K . Thus $P_w \widehat{X}^o = 0$; since by (3.6), $P_w = I_{2\ell-1} - G_{\Phi, \Phi^*, t_o}$ the identities (3.17) and (3.18) imply that $X^{o*} X^o - X^{o*} S X^o = 0$. Hence, since the matrix $I - S$ is Hermitian and the associated quadratic form is positive semidefinite, $X^o - S X^o = 0$, i.e. 1 is an eigenvalue of S .

Theorem 3.5. *Suppose that the assumptions of Theorem 2.1 are satisfied with $h \neq \omega/\ell$ and let $\mathcal{I} \subset \mathbb{Z}$ be a finite assigned set. If the components of the vector W are $P_{\mathcal{I}, h}$ -independent on $I_- \cup I_+$, then it is possible to recover the samples $f_j(nt_o)$, $n \in \mathcal{I}$, $1 \leq j \leq 2\ell$.*

Proof. Let $\mathcal{I} = \{\ell_1, \ell_2, \dots, \ell_N\}$. As in the proof of the previous theorem, we write the system as in (3.16), where the block matrix S is now

$$S = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{12\ell} \\ S_{21} & S_{22} & \cdots & S_{22\ell} \\ \vdots & \vdots & \ddots & \vdots \\ S_{2\ell 1} & S_{2\ell 2} & \cdots & S_{2\ell 2\ell} \end{bmatrix}$$

and the entries of the submatrices $S_{k,j}$ $k, j = 1, \dots, 2\ell$ are as in (3.15). By arguing as in the proof of (3.17) and using (3.8) instead of (3.6), we get

$$X^* X - X^* S X = \frac{1}{h} \int_{I_- \cup I_+} \left\| \left(I_{2\ell} - G_{\Phi, \Phi^*, t_o}(t) \right) \widehat{X}(t) \right\|_{\mathbb{C}^{2\ell}}^2 dt$$

for all $X \in \mathbb{C}^{N2\ell}$. The conclusion follows by the same arguments of the previous theorem. \square

Next we consider the case where the samples to reconstruct concern a number λ of generators less than the length of the space B_ω . In Corollaries 3.6 and 3.7 below we find conditions such that reconstruction is possible. By renaming the generators, we can assume that the missing samples are relative to the first λ generators.

Corollary 3.6. *Assume that the assumptions of Theorem 2.2 are satisfied with $h \neq \omega/(\ell - 1/2)$. Let $\mathcal{I} \subset \mathbb{Z}$ be a finite set and let λ be such that $1 \leq \lambda < 2\ell - 1$. If the components of index $1, 2, \dots, \lambda$ of the vector W are $P_{\mathcal{I}, h}$ -independent on K , then it is possible to recover the samples $f_j(nt_o)$, $n \in \mathcal{I}$, $1 \leq j \leq \lambda$.*

Proof. Let $\mathcal{I} = \{\ell_1, \ell_2, \dots, \ell_N\}$. Starting from (3.13) and moving to the right hand side the terms containing the known samples, we obtain

$$\begin{aligned} f_k(\ell_m t_o) - \sum_{j=1}^{\lambda} \sum_{p=1}^N f_j(\ell_p t_o) (\varphi_j^* * \tilde{\varphi}_k)(\ell_m t_o - \ell_p t_o) \\ = \sum_{j=1}^{\lambda} \sum_{n \notin \mathcal{I}} f_j(nt_o) (\varphi_j^* * \tilde{\varphi}_k)(\ell_m t_o - nt_o) \\ + \sum_{j=\lambda+1}^{2\ell-1} \sum_{n \in \mathbb{Z}} f_j(nt_o) (\varphi_j^* * \tilde{\varphi}_k)(\ell_m t_o - nt_o). \end{aligned}$$

Denoting by $b_k(m)$ the right hand side we get

$$(3.19) \quad f_k(\ell_m t_o) - \sum_{j=1}^{\lambda} \sum_{p=1}^N f_j(\ell_p t_o) (\varphi_j^* * \tilde{\varphi}_k)(\ell_m t_o - \ell_p t_o) = b_k(m)$$

$1 \leq k \leq \lambda$, $1 \leq m \leq N$. As in the proof of Theorem 3.4 we regard these equations as a system in the unknowns $f_k(\ell_m t_o)$, $1 \leq m \leq N$, $1 \leq k \leq \lambda$. To write it in matrix form we denote the unknowns by $z_k(m)$ and we write

$$\mathbf{z}_k = (z_k(1), \dots, z_k(N)) \quad \mathbf{b}_k = (b_k(1), \dots, b_k(N)) \quad 1 \leq k \leq \lambda$$

and

$$Z = (\mathbf{z}_1, \dots, \mathbf{z}_\lambda) \quad B = (\mathbf{b}_1, \dots, \mathbf{b}_\lambda).$$

Then the equations in (3.19) can be written

$$(3.20) \quad (I - S^{[\lambda]}) Z = B,$$

where $S^{[\lambda]}$ is the block matrix obtained from the matrix S in (3.14) by eliminating the blocks $S_{j,k}$ with j and k greater than λ , I is the $N\lambda \times N\lambda$ identity matrix, and Z and B are in $\mathbb{C}^{N\lambda}$. It is possible to recover the missing samples if (3.20) can be solved for all B and this happens if and only if the coefficient matrix $I - S^{[\lambda]}$ is invertible, i.e. if and only if 1 is not an eigenvalue of $S^{[\lambda]}$. As in the proof of Theorem 3.4, we shall show that, if 1 is an eigenvalue of $S^{[\lambda]}$, then the first λ components of the vector W are $P_{\mathcal{I},h}$ -independent on K .

Denote by Q the projection on $\mathbb{C}^{N(2\ell-1)}$ mapping the vector $X = (\mathbf{x}_1, \dots, \mathbf{x}_{2\ell-1})$ to $QX = (\mathbf{x}_1, \dots, \mathbf{x}_\lambda, 0, \dots, 0)$. We identify the operator Q with the matrix

$$\left[\begin{array}{c|c} I_\lambda & 0 \\ \hline 0 & 0 \end{array} \right]$$

where I_λ is the $\lambda \times \lambda$ identity matrix. We observe that 1 is an eigenvalue of $S^{[\lambda]}$ iff it is an eigenvalue of QS with eigenvector $X_o \in \text{Ran}(Q)$.

Next assume that 1 is an eigenvalue of the matrix $S^{[\lambda]}$. Then 1 is an eigenvalue of QS and there exists a non-null $X_o \in \text{Ran}(Q)$ such that $X_o - QSX_o = 0$; i.e. $QX_o - QSQX_o = 0$. Hence $X_o^* QX_o - X_o^* QSQX_o = 0$. This can be written

$$(QX_o)^* QX_o - (QX_o)^* S QX_o = 0.$$

By applying (3.17) with $X = QX_o$ we get

$$(I_{2\ell-1} - G_{\Phi, \Phi^*, t_o})(\widehat{QX_o}) = 0 \quad \text{a.e. in } K.$$

By (3.6) it follows that $P_W(\widehat{QX_o^T}) = 0$, i.e. $\widehat{QX_o}(t) = Q\widehat{X_o}(t)$ is orthogonal to $W(t)$ for a.e. t in K . This shows that

$$\sum_{j=1}^{\lambda} \widehat{\mathbf{x}_{o_j}}(t) W_j(t) = 0 \quad \text{for a.e. } t \text{ in } K.$$

We have thus proved that if 1 is an eigenvalue of $S^{[\lambda]}$, then the first λ components of W are $P_{\mathcal{I},h}$ -dependent on K . \square

Corollary 3.7. *Suppose that the assumptions of Theorem 2.1 are satisfied with $h \neq \omega/\ell$. Let \mathcal{I} be an assigned finite set and λ be such that $1 \leq \lambda < 2\ell$. If the components of index $1, 2, \dots, \lambda$ of the vector W are $P_{\mathcal{I},h}$ -independent on the interval $I_- \cup I_+$, then it is possible to recover the samples $f_j(nt_o)$ $n \in \mathcal{I}$, $1 \leq j \leq \lambda$.*

We omit the proof since it is quite similar to that of Corollary 3.6.

4. RECOVERY OF MISSING SAMPLES IN THE DERIVATIVE SAMPLING

This section is dedicated to derivative sampling and to the recovery of missing samples in the corresponding reconstruction formulas. Fix an integer $L > 1$ and let $\Phi_L = (\varphi_1, \dots, \varphi_L)$ be defined by

$$(4.1) \quad \widehat{\varphi_1} = \chi_{[-\omega, \omega]}, \quad \widehat{\varphi_2} = ix\chi_{[-\omega, \omega]}, \quad \dots, \widehat{\varphi_L} = (ix)^{L-1}\chi_{[-\omega, \omega]}.$$

It is well known that if $h = 2\omega/L$, i.e. if $t_o = \pi L/\omega$, the family E_{Φ_L, t_o} is a Riesz basis for B_ω and in the reconstruction formula for a function $f \in B_\omega$ the coefficients are the values of f and its first $L-1$ derivatives at the sample points (see for example [HS], [Hi]).

In the first part of this section we show that E_{Φ_L, t_o} is a frame for B_ω for any $h \in [2\omega/L, 2\omega/(L-1))$ i.e. for any $t_o \in ((L-1)\pi/\omega, L\pi/\omega]$ (see Theorem 4.2). We need a lemma. Denote by M_n the $n \times (n+1)$ matrix

$$(4.2) \quad M_n(x) = \begin{bmatrix} 1 & \tau_h(ix) & \tau_h(ix)^2 & \dots & \tau_h(ix)^n \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \tau_{nh}(ix) & \tau_{nh}(ix)^2 & \dots & \tau_{nh}(ix)^n \end{bmatrix}.$$

Lemma 4.1. *Let $M_{n,r}$, $r = 0, 1, \dots, n$ be the submatrix of M_n obtained by suppressing the r -th column. Then the determinant of $M_{n,r}$ is a polynomial of degree $n-r$.*

Proof. First we observe that $M_{n,n}$ is a Vandermonde matrix. Thus

$$(4.3) \quad \det M_{n,n} = \prod_{1 \leq k < j \leq n} (\tau_{jh}(ix) - \tau_{kh}(ix)) = (ih)^n \prod_{1 \leq k < j \leq n} (j-k) = (ih)^n \prod_{p=1}^{n-1} p!$$

is a constant. Next we show that

$$(4.4) \quad (\det M_{n,r})' = i(r+1) \det M_{n,r+1} \quad r = 0, \dots, n-1.$$

We recall that the derivative of the determinant of a $n \times n$ matrix is the sum of n determinants, each obtained by replacing in the matrix the elements of a column by their derivatives. First we prove formula (4.4) for $r = n-1$, i.e. for the matrix

$$(4.5) \quad M_{n,n-1} = \begin{bmatrix} 1 & \tau_h(ix) & \dots & \tau_h(ix)^{n-2} & \tau_h(ix)^n \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \tau_{nh}(ix) & \dots & \tau_{nh}(ix)^{n-2} & \tau_{nh}(ix)^n \end{bmatrix}.$$

We observe that the derivative of the entries of the first column is zero and that for $j = 1, 2, \dots, n-2$ the derivative of the j -th column is a multiple of the preceding column. Hence the determinants of the first $n-1$ matrices vanish. It remains only the matrix obtained by derivating the last column. Therefore

$$(\det M_{n,n-1})' = \begin{bmatrix} 1 & \tau_h(ix) & \dots & \tau_h(ix)^{n-2} & in\tau_h(ix)^{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \tau_{nh}(ix) & \dots & \tau_{nh}(ix)^{n-2} & in\tau_{nh}(ix)^{n-1} \end{bmatrix} = in \det M_{n,n}.$$

Hence we have proved formula (4.4) for $r = n - 1$; the proof for $r = 0, 1, \dots, n - 2$ is similar. By (4.3) and (4.4) the lemma follows. \square

Theorem 4.2. *Let $\Phi_L = (\varphi_1, \varphi_2, \dots, \varphi_L)$ be as in (4.1). If t_o is a positive number such that $(L - 1)\pi/\omega < t_o \leq L\pi/\omega$, then E_{Φ_L, t_o} is a frame for B_ω . If $t_o = L\pi/\omega$ then E_{Φ_L, t_o} is a Riesz basis for B_ω .*

Proof. We shall prove the theorem only in the case L odd. The proof in the case L even is similar. If $L = 2\ell - 1$, then $\omega/(\ell - \frac{1}{2}) \leq h < \omega/(\ell - 1)$, since $h = 2\pi/t_o$. We show that the assumptions in Theorem 2.2 are all satisfied. Indeed, since $\sum_{j=1}^{2\ell-1} |\widehat{\varphi}_j|^2 = \sum_{j=0}^{2\ell-1} x^{2j}$, condition (2.13) is satisfied. Next we prove that there exists a positive number σ such that (2.14) holds. By (2.12), in the interval K the matrix obtained by deleting the vanishing row in the pre-Gramian is the $(2\ell - 2) \times (2\ell - 1)$ matrix

$$(4.6) \quad \mathbb{J}_{\Phi^*, t_o}(x) = \sqrt{h} \begin{bmatrix} 1 & \tau_{-(\ell-1)h}(ix) & \dots & \tau_{-(\ell-1)h}(ix)^{2\ell-2} \\ \vdots & \vdots & & \vdots \\ 1 & \tau_{(\ell-2)h}(ix) & \dots & \tau_{(\ell-2)h}(ix)^{2\ell-2} \end{bmatrix}$$

i.e.

$$(4.7) \quad \mathbb{J}_{\Phi^*, t_o}(x) = \sqrt{h} \tau_{\ell h} M_{2\ell-2}(x).$$

By (4.3) the $(2\ell - 2) \times (2\ell - 2)$ minor of \mathbb{J}_{Φ^*, t_o} obtained by suppressing the last column is

$$h^{\ell-1} \det M_{2\ell-2, 2\ell-2}(x) = (-1)^{\ell-1} h^{3\ell-3} \prod_{p=1}^{2\ell-2} p! \quad \forall x \in \mathbb{R}.$$

We recall that $[J_{\Phi, t_o}]_{2\ell-2}$ is the sum of the squares of the absolute values of the minors of order $2\ell - 2$; hence

$$[J_{\Phi, t_o}]_{2\ell-2} \geq h^{6(\ell-1)} \left(\prod_{p=1}^{2\ell-2} p! \right)^2 > 0 \quad \text{in } K.$$

This shows that (2.14) is satisfied. Next we consider the interval K_- ; here by (2.12) and (4.1)

$$J_{\Phi, t_o} = \sqrt{h} \tau_{-\ell h} M_{2\ell-1, 2\ell-1}.$$

By (4.3) its determinant is

$$\det J_{\Phi, t_o}(x) = (-1)^{\ell+1} i h^{3\ell-3/2} \prod_{p=1}^{2\ell-1} p! \quad \forall x \in K_-.$$

The same formula holds in K_+ . This proves (2.15). Thus the assumptions of Theorem 2.2 are satisfied. This concludes the proof for the case $\omega/\ell < h < \omega/(\ell - \frac{1}{2})$. If $h = \omega/(\ell - 1/2)$, then the interval K is empty. By arguing as before, one can show that (2.13) and (2.15) hold on the intervals K_- and K_+ . Hence, by Theorem 2.2, E_{Φ, t_o} is a Riesz basis. \square

Denote by $\Phi_L^* = (\varphi_1^*, \dots, \varphi_L^*)$ the canonical dual generators of the frame E_{Φ_L, t_o} . By Theorem 4.2, every $f \in B_\omega$ can be expanded in terms of the elements of the dual family as in (3.1). By (4.1) the coefficients of this sum are

$$f_j(nt_o) = \sqrt{2\pi} (-1)^{j-1} f^{(j-1)}(nt_o) \quad n \in \mathbb{Z} \quad j = 1, \dots, L.$$

Hence

$$(4.8) \quad f(x) = \sqrt{2\pi} \sum_{j=1}^L \sum_{n \in \mathbb{Z}} (-1)^{j-1} f^{(j-1)}(nt_o) \varphi_j^*(x - nt_o) \quad \forall f \in B_\omega,$$

where $L = 2\ell$ if $\omega/\ell \leq h < \omega/(\ell-1/2)$ and $L = 2\ell-1$ if $\omega/(\ell-1/2) \leq h < \omega/(\ell-1)$. We shall call (4.8) the *derivative oversampling formula of order $L-1$* .

Proposition 4.4 below shows that, for band-limited signals, it is possible to recover any finite set of missing samples in the derivative sampling expansion (4.8). Thus we generalize to arbitrary order the result of Santos and Ferreira in [SF]. We need a lemma. Denote by ∂P the degree of a polynomial P .

Lemma 4.3. *Let P_i , $i = 1, 2, \dots, N$, be non-null polynomials such that*

$$\partial P_1 > \partial P_2 > \dots > \partial P_N.$$

If $\sum_{j=1}^N \gamma_j P_j = 0$ for some continuous h -periodic functions γ_i , $i = 1, 2, \dots, N$, then $\gamma_i = 0$ for all $i = 1, 2, \dots, N$.

Proof. The case $N = 1$ is obvious; assume by induction that the statement is true for N polynomials. Let P_i , $i = 1, 2, \dots, N+1$, be non-null polynomials such that

$$\partial P_1 > \partial P_2 > \dots > \partial P_N > \partial P_{N+1}$$

and

$$\gamma_1 P_1 + \gamma_2 P_2 + \dots + \gamma_N P_N + \gamma_{N+1} P_{N+1} = 0.$$

Denote by q be the degree of P_1 . By dividing by x^q and taking the limit to infinity, we obtain that γ_1 is identically zero. Hence $\sum_{j=2}^{N+1} \gamma_j P_j = 0$. The conclusion follows, by the inductive assumption. \square

Proposition 4.4. *Let $\Phi_L = (\varphi_1, \varphi_2, \dots, \varphi_L)$ the frame generators defined by (4.1). If $(L-1)\pi/\omega < t_o < L\pi/\omega$, then in the sampling formula (4.8) it is possible to recover the missing samples*

$$f^{(j)}(nt_o) \quad n \in \mathcal{I}, \quad 1 \leq j \leq L-1,$$

for any finite set $\mathcal{I} \subset \mathbb{Z}$.

Proof. We shall only prove the theorem when L is odd. The proof for L even is similar. Let $L = 2\ell - 1$; then $\omega/(\ell - 1/2) \leq h < \omega/(\ell - 1)$, by the assumption on t_o . We recall from Section 3 that W denotes the cross product of the rows of $h^{-\frac{1}{2}} \mathbb{J}_{\Phi, t_o}$; thus, up to a sign, its components W_k are the minors of order $2\ell - 2$ of $h^{-\frac{1}{2}} \mathbb{J}_{\Phi, t_o}$. By Theorem 3.4 it suffices to prove that the components of the vector W are $P_{\mathcal{I}, h}$ -independent on K for any finite set \mathcal{I} . By (4.7) and Lemma 4.1, W_k is a polynomial of degree $2\ell - 1 - k$ for $k = 1, 2, \dots, 2\ell - 1$. Thus

$$\partial W_1 > \partial W_2 > \dots > \partial W_{2\ell-1}.$$

Let now $p_1, p_2, \dots, p_{2\ell-1}$ be trigonometric polynomials in $P_{\mathcal{I}, h}$ such that

$$\sum_{j=1}^{2\ell-1} p_j(t) W_j(t) = 0, \quad t \in K.$$

By analytic continuation, this identity holds on \mathbb{R} . Thus, by Lemma 4.3, $p_j = 0$ for all $j = 1, \dots, 2\ell - 1$. This concludes the proof of the proposition. \square

5. THE FIRST ORDER FORMULA: EXPLICIT DUALS AND NUMERICAL EXPERIMENTS

Obviously in applications, to use the reconstruction formula (4.8), it is necessary to find the dual generators $\varphi_1^*, \dots, \varphi_L^*$. When $L = 2, 3$ we gave an explicit expression of the Fourier transforms of the duals in terms of the Fourier transforms of the generators in (see Corollaries 5.2 and 5.3 in [DP]). For greater values of L one can use the general formulas given in Theorems 4.6 and 4.7 in [DP]. In this section we provide an explicit expression of the duals for the first-order derivative formula ($L = 2$) (see (5.2)) and we present a numerical experiment of recovery of missing samples and reconstruction of a signal.

Let φ_1, φ_2 be the functions defined in (4.1) for $L = 2$:

$$\widehat{\varphi_1} = \chi_{[-\omega, \omega]} \quad \widehat{\varphi_2} = ix\chi_{[-\omega, \omega]}.$$

By Theorem 4.2, E_{Φ, t_o} is a frame for B_ω if $\omega \leq h < 2\omega$; if $h = \omega$ then it is a Riesz basis for B_ω . The sampling formula (4.8) reduces to

$$(5.1) \quad f = \sqrt{2\pi} \sum_{k \in \mathbb{Z}} f(kt_o) \tau_{-kt_o} \varphi_1^* - f'(kt_o) \tau_{-kt_o} \varphi_2^*.$$

The Fourier transforms of the dual generators are

$$\widehat{\varphi_1^*}(x) = \begin{cases} \frac{1}{h} \left(1 - \frac{|x|}{h}\right) & h - \omega < |x| < \omega \\ \frac{1}{h(1+x^2)} & |x| < h - \omega \end{cases} \quad \widehat{\varphi_2^*}(x) = \begin{cases} \frac{i}{h^2} \text{sign}(x) & h - \omega < |x| < \omega \\ \frac{ix}{h(1+x^2)} & |x| < h - \omega \end{cases}$$

(see Corollary 5.2 in [DP]). Figure 1 shows $\widehat{\varphi_1^*}, \widehat{\varphi_2^*}$ for $h = 1.6\omega$.

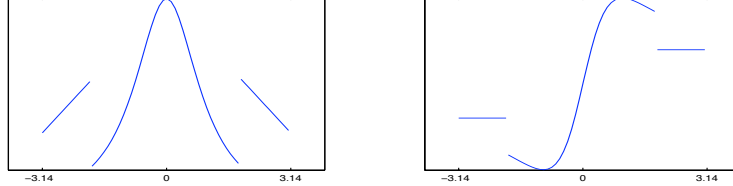


Figure 1: The Fourier transforms of the duals for $h = 1.6\pi$.

By Fourier inversion

$$(5.2) \quad \begin{aligned} \varphi_1^*(x) = & \frac{1}{\sqrt{2\pi}} \left[\frac{2\omega}{h} H(\text{sinc}(\omega x) - \text{sinc}(Hx)) + \frac{2}{h^2 x^2} (\cos(Hx) - \cos(\omega x)) \right. \\ & \left. + \frac{1}{h} \int_{|t| < H} \frac{1}{1+t^2} e^{itx} dt \right] \end{aligned}$$

$$\varphi_2^*(x) = \frac{1}{\sqrt{2\pi}} \left[\frac{2}{h^2} (\cos(\omega x) - \cos(Hx)) x^{-1} + \frac{i}{h} \int_{0 < |t| < H} \frac{t}{1+t^2} e^{itx} dt \right]$$

where $H = h - \omega$ and $\text{sinc} = \sin(x)/x$. The integrals in the right hand side of this formula can be written as linear combinations of sine and cosine integrals, so that

it is possible to implement the duals without any approximation. Observe that if $h = \omega$ then

$$\varphi_1^*(x) = \frac{1}{\sqrt{2\pi}} \text{sinc}^2\left(\frac{\omega x}{2}\right) \quad \varphi_2^*(x) = -\frac{1}{\sqrt{2\pi}} x \text{sinc}^2\left(\frac{\omega x}{2}\right).$$

Following Ferreira, we have chosen the test function

$$(5.3) \quad f_o(x) = \text{sinc}(\pi(x - 2.1)) - 0.7 \text{sinc}(\pi(x + 1.7))$$

plotted in Figure 2. In general, redundancy makes frame expansions robust to

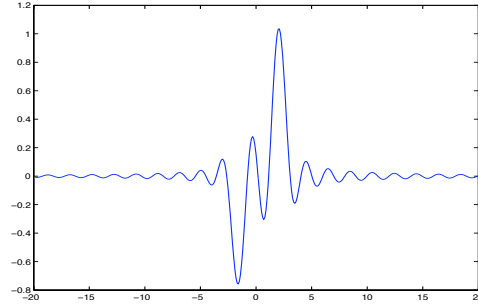


Figure 2: The function f_o in $[-20, 20]$.

errors like noise, quantization and losses. However, for this derivative frame, even the reconstruction of a signal where only two samples are missing may lead to poor results. To illustrate this point, we have reconstructed the signal f_o in (5.3) by using formula (5.1), when the samples $f_o(4t_o)$ and $f_o'(4t_o)$ are set to zero:

$$f_o^* = \sqrt{2\pi} \sum_{k \neq 4} f_o(kt_o) \tau_{-kt_o} \varphi_1^* - f_o'(kt_o) \tau_{-kt_o} \varphi_2^*.$$

Figure 3 shows the functions f_o^* and f_o in the interval $[-10, 10]$.

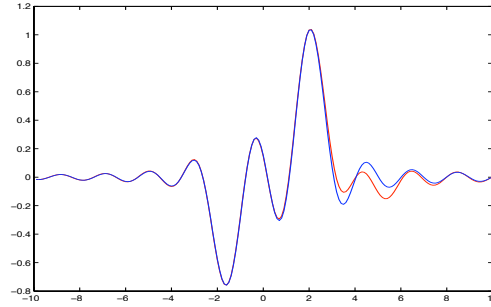


Figure 3: The functions f_o^* and f_o in $[-10, 10]$.

In our experiment we assume that 10 samples of f_o and f_o' are unknown and we recover them by solving the 20×20 system (3.16). Then we obtain the reconstructed function f_o^c via formula (5.1), where the missing samples are replaced by

the computed ones. We suppose that the missing samples are $\{f_o(\ell_k t_o), f_o'(\ell_k t_o), k = 1, 2, \dots, 10\}$, where $\ell_k = -16 + 3(k - 1)$. We have obtained the computed samples with an error of order 10^{-4} and a relative error of order 10^{-2} ; the graph of the reconstructed function f_o^c is undistinguishable from that of f_o .

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